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THE DETERMINATION OF A STABILITY INDICATIVE FUNCTION FOR LINEAR SYSTEMS WITH MULTIPLE DELAYS*

By John D. Shaughnessy and Yasutada Kashiwagi

SUMMARY

A theoretical study is made of the stability of a class of linear differential-difference equations with multiple delays. A direct method for determining the exact stability boundaries for homogeneous, linear differential-difference equations with constant coefficients and constant delays is formulated. This formulation results in a stability indicative function, depending on a single parameter, which determines the number of roots of the transcendental characteristic equation that have positive real parts. It is proved that the system is stable if and only if this function has a value of zero.

A second-order system with delays in the velocity and position feedback terms is considered as an example, and the stability regions for this system are determined for a range of delays and coefficients. It is observed that introduction of a delay has a definite destabilizing effect on the system, and introduction of a second delay has a compounding effect to further reduce stability. However, this example clearly illustrates that certain combinations of delays can stabilize an unstable system. This phenomenon is discussed from a theoretical point of view.

INTRODUCTION

Time delays or retarded actions are present in modern problems of guidance and control and the dynamics of manned and unmanned space vehicles. The effect of such delays is the subject of this paper. These phenomena may occur in several different ways; for example, in remote control of distant space vehicles the communications delay can adversely affect the stability of the overall control system. (See ref. 1.) Time delays in the engine response of large jet transports can seriously affect the handling qualities of the aircraft. In manned systems, lagging commands caused by slow human response can cause a normally stable system to become unstable. Hypervelocity entry

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*Most of this material was included in a thesis entitled "Stability of Linear Systems With Multiple Delays" submitted by John D. Shaughnessy (with Yasutada Kashiwagi as thesis adviser) in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering, Virginia Polytechnic Institute, May 1967.

vehicles can lose aerodynamic stability as a result of flow-field lags caused by the spacecraft motions and ablation. (See ref. 2.) As a final example, combustion delays in rocket motors can lead to erratic or intermittent running, possibly terminating with an explosion of the motor. (See ref. 3.) References 4 and 5 are comprehensive bibliographies of works published before 1960 that pertain to time-lag systems. Reference 6 contains a general discussion of the existence and stability of solutions of differential-difference equations.

As of this writing there is no straightforward or practical method for determining the stability characteristics of linear systems with multiple time delays. However, it was suggested in reference 7 that the stability indicative function for linear systems with a single delay, introduced in reference 8, could be extended to include systems with multiple delays. The subject of this report is the extension of the stability indicative function to include such systems.

The mathematics of delay problems often results in the so-called differential-difference equations, which are similar in several respects to ordinary differential equations; however, there are significant dissimilarities. It is easy to show that a linear differential-difference equation with constant coefficients is formally like an ordinary differential equation of infinite order. Thus, the characteristic polynomial for the differential-difference equation obtained by using the classical Laplace transformation will be of infinite degree and will necessarily have a countably infinite number of characteristic roots. Another important distinction between ordinary differential equations and differential-difference equations is in the initial conditions. For the differential-difference equations, initial functions rather than initial constants must be specified.

As will be shown, the stability of a large class of linear time-delay problems may be obtained by determining whether any roots of the characteristic equation have positive real parts. Even though these systems have an infinite number of characteristic roots, it is possible to determine how many of these roots have positive real parts. In this paper a method will be formulated for finding the stability of these systems in terms of the time delays and the coefficients of the differential-difference equations.

SYMBOLS

A	real, positive constant
a_n,b_n	real constants
f_1,f_2	functions of ω defined by equations (49) and (50)

H_1,H_2	functions of s defined by equation (5)
I	stability indicative function
j	positive integer
k	positive integer
L	characteristic function
L_R	real part of $L(\sigma+i\omega)$
L_I	imaginary part of $L(\sigma+i\omega)$
l	positive integer
M,m	positive integers
min	minimum
N,n	positive integers
p	positive integer
s	Laplace transform variable
T_p	position time delay
T_v	velocity time delay
t	nondimensional time
x	dependent variable
z	positive integer
α	particular value of σ
β	particular value of ω

ζ	real constant
Θ	set of critical values θ^*'s
$\theta_0, \theta_1, \ldots$	nondimensional time delays
$\theta_0^*, \theta_1^*, \ldots$	critical values of $\theta_0, \theta_1, \ldots$
σ	real part of s
τ	time
Ω	real constant
ω	imaginary part of s
ω^*	critical value of ω

Mathematical notation:

$\arg s$	argument of s
$[a,b]$	closed interval $a \leqq t \leqq b$
$[a,b)$	semiclosed interval $a \leqq t < b$
i	$\sqrt{-1}$
$\operatorname{sgn} x$	sign of x: $x/\|x\|$
$[x\|P]$	set of x's having property P
$x^{(n)}(t)$	nth derivative of x with respect to t
Δ	Jacobian determinant
ϵ	is an element of

A bar over a symbol denotes the prescribed value.

STABILITY OF LINEAR SYSTEMS WITH MULTIPLE DELAYS

The following form of homogeneous linear differential-difference equations with constant coefficients and constant delays will be considered:

$$a_N x^{(N)}(t) + b_N x^{(N)}\left(t - \theta_N\right) + a_{N-1} x^{(N-1)}(t) + b_{N-1} x^{(N-1)}\left(t - \theta_{N-1}\right)$$

$$+ \ . \ . \ . + a_0 x(t) + b_0 x\left(t - \theta_0\right) = 0 \tag{1}$$

Equation (1) may be expressed more compactly as

$$\sum_{n=0}^{N}\left[a_n x^{(n)}(t) + b_n x^{(n)}\left(t - \theta_n\right)\right] = 0 \tag{2}$$

The coefficients a_n and b_n are real constants and the constant delays θ_n are nonnegative finite values and are not necessarily distinct. By using the Laplace transformation, the characteristic equation of equation (2) is found to be

$$\sum_{n=0}^{N}\left(a_n s^n + b_n s^n e^{-\theta_n s}\right) = 0 \tag{3}$$

where $s = \sigma + i\omega$ (σ and ω are real variables).

Pontryagin (ref. 9) has shown that if $a_N = 0$, $b_N \neq 0$, and $\theta_N > 0$, the system is unstable. In reference 10 it is shown that if $a_N \neq 0$, $b_N \neq 0$, and $\theta_N > 0$, the system may be unstable even though all characteristic roots of equation (3) have negative real parts. These two cases will not be considered; rather, an examination will be made of the case where $a_N \neq 0$ and $b_N = 0$, written as

$$L(s) = \sum_{n=0}^{N} a_n s^n + \sum_{n=0}^{N-1} b_n s^n e^{-\theta_n s} = 0 \qquad \left(a_N \neq 0\right) \tag{4}$$

Because of the infinite-degree nature of equation (4), exact determination of the characteristic roots is obviously impossible; however, the number of roots with positive real parts can be determined. Theorem 1 of the appendix can then be employed to determine the stability of the system. If stability boundaries are desired, the number of roots with positive real parts is determined as a function of the parameters of interest.

Before entering the development of this idea, several existing stability criteria are considered. The Routh-Hurwitz stability criterion clearly cannot be used with differential-difference equations. The Nyquist criterion can be used with only limited success because, as pointed out in reference 8, there is a question regarding the order of the characteristic equation which may lead to contradictory results. Also, stability boundaries would be quite difficult to obtain. Pontryagin (ref. 9) gives necessary and sufficient conditions for stability of linear systems with delays, but the delays must be integral multiples of each other. Finally, the Lyapunov stability criterion is applicable to linear as well as nonlinear differential-difference equations (see refs. 6, 11, and 12); however, the resulting stability boundaries depend on the Lyapunov function used and oftentimes do not give the largest region of stability.

It was pointed out above that, in general, each characteristic root of equation (4) cannot be located, but the number of roots having positive real parts can be determined. Further, by theorem 1 of the appendix, the system is stable if, and only if, this number is zero. Before developing the method for calculating the number of roots that lie in the right half of the s-plane, some important aspects of this modified root-locus technique, extended to delay systems and first presented in reference 8, should be mentioned.

Considerable information can be realized by studying a periodic solution. When some of the roots (nonmultiple) of equation (4) lie on the imaginary axis and the rest lie in the left half-plane, a periodic solution will be obtained. These pure imaginary roots correspond to intersections of the root loci of equation (4) with the imaginary axis. The root loci are the plots of the variations of the roots of equation (4) with changes in a single parameter, say an arbitrary delay θ_k, from zero to its maximum value, while all other delays are fixed. The values of the parameter that correspond to an intersection of a root locus with the imaginary axis will be denoted as critical values θ_k^*. Also a point $(0,i\omega^*)$ in the s-plane will be denoted as a critical point. It will be shown that, in general, the number of critical values will be zero or countably infinite and these critical values can be found by replacing s with $i\omega^*$ in equation (4). If there are an infinite number of critical values, say $\theta_{k,1}^* < \theta_{k,2}^* < \ldots < \theta_{k,l}^* < \ldots$ (where the subscripts $1, 2, \ldots, l, \ldots$ indicate the different critical values of θ_k), then the stability of the system remains unchanged in the interval $0 \leqq \theta_k < \theta_{k,1}^*$ but is unknown for $\theta_k \geqq \theta_{k,1}^*$ because, so far, nothing is known about the behavior of the root loci at the intersections at $s = i\omega^*$.

The statements regarding the behavior of the root loci near the critical points will now be considered in more detail. This analysis follows directly from reference 8. A system which is stable when $\theta_k = 0$ is now considered, and it is supposed that there are two critical values $\theta^*_{k,1} < \theta^*_{k,2}$ for $\theta_k \epsilon \left[0,A\right]$ where A is a real, positive constant. Thus the system is stable for $0 \leqq \theta_k < \theta^*_{k,1}$, but the stability is not known for $\theta_k \geqq \theta^*_{k,1}$.

There are several ways in which the root locus might intersect the imaginary axis when $\theta_k = \theta^*_{k,1}$ and $\theta_k = \theta^*_{k,2}$ for $\theta_k \epsilon \left[0,A\right]$. Figure 1 illustrates three possible cases. Just the upper half of the s-plane is considered, since the loci are symmetric with respect to the real axis. If the loci have the form in figure 1(a), the system is stable for $0 \leqq \theta_k \leqq \theta^*_{k,1}$ and unstable for $\theta^*_{k,1} < \theta_k \leqq A$. In the case of figure 1(b), the system is stable for $0 \leqq \theta_k \leqq \theta^*_{k,1}$ and $\theta^*_{k,2} \leqq \theta_k \leqq A$ but unstable for $\theta^*_{k,1} < \theta_k < \theta^*_{k,2}$. In figure 1(c), the system is stable for $0 \leqq \theta_k \leqq A$. Even if all three cases have the same critical values $\theta^*_{k,1}$ and $\theta^*_{k,2}$, these cases represent different stability situations for $\theta^*_{k,1} < \theta_k \leqq A$. Thus it is clear that in order to investigate the stability of systems described by equation (4), it is not sufficient to find only the critical values of a parameter. If only the critical values $\theta^*_{k,1}, \theta^*_{k,2}, \ldots, \theta^*_{k,l}, \ldots$ are known for the given system, then all that can be said is that the system has the same stability characteristics for $0 \leqq \theta_k < \theta^*_{k,1}$ $\left(\text{where } \theta^*_{k,1} < \theta^*_{k,2} < \ldots < \theta^*_{k,l} < \ldots\right)$ as for $\theta_k = 0$. Thus it is entirely possible for delays to make stable systems unstable and unstable systems stable. In general, if the stability of a system with delays is to be determined for $\theta_k \epsilon \left[\theta^*_{k,1},\infty\right)$, further information such as root loci directions at the critical points must be obtained.

Calculation of the critical values and determination of the directions of the root loci at critical points are now considered. The characteristic equation can be written in the form

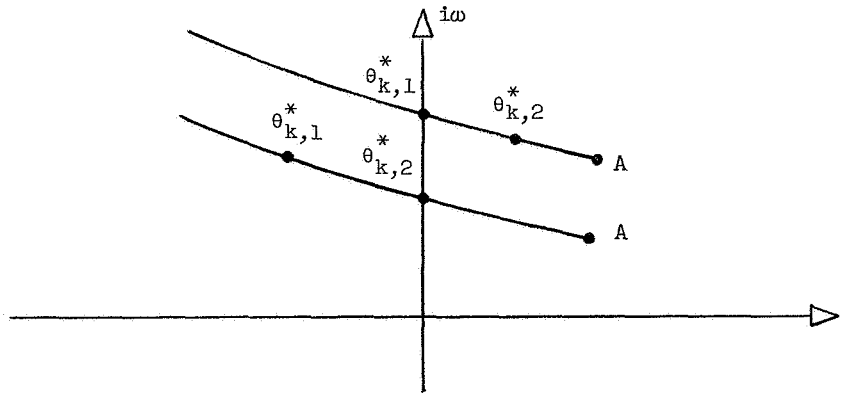
$$L(s) = H_1(s) - H_2(s)e^{-\theta_k s} = 0 \tag{5}$$

where

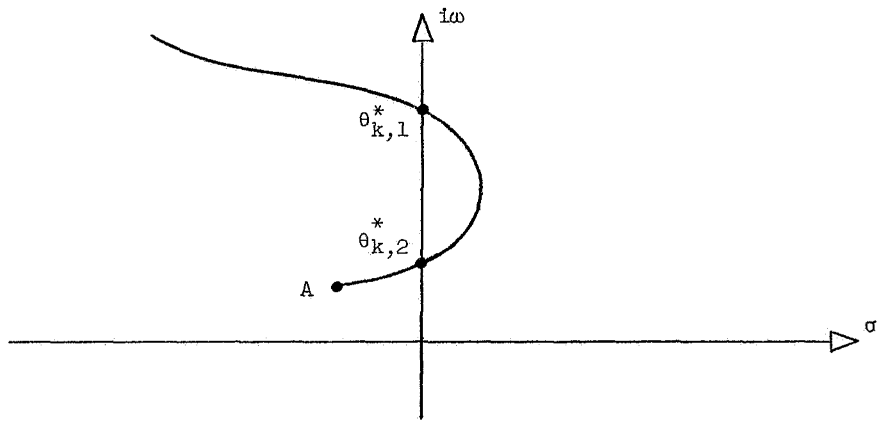
$$H_1(s) = \sum_{n=0}^{N} a_n s^n + \sum_{n=0}^{k-1} b_n s^n e^{-\theta_n s} + \sum_{n=k+1}^{N-1} b_n s^n e^{-\theta_n s}$$

and

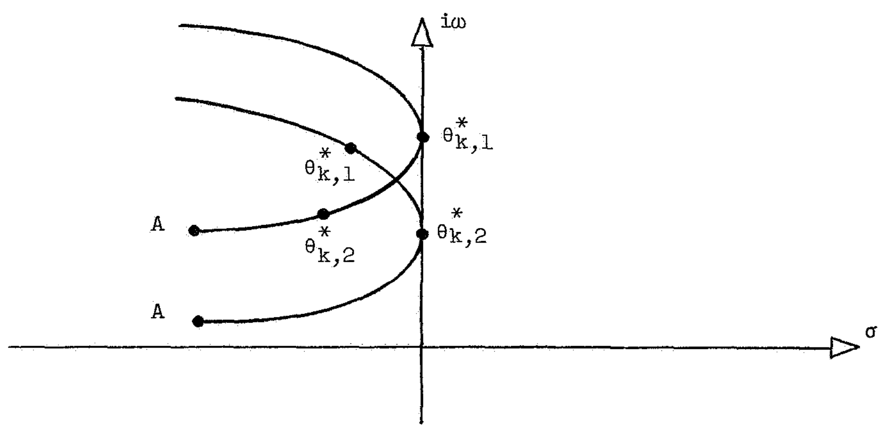
$$H_2(s) = -b_k s^k$$



(a) System stable for $0 \leqq \theta_k \leqq \theta^*_{k,1}$ and unstable for $\theta^*_{k,1} < \theta_k \leqq A$.



(b) System stable for $0 \leqq \theta_k \leqq \theta^*_{k,1}$ and $\theta^*_{k,2} \leqq \theta_k \leqq A$ but unstable for $\theta^*_{k,1} < \theta_k < \theta^*_{k,2}$.



(c) System stable for $0 \leqq \theta_k \leqq A$.

Figure 1.- Behavior of root loci in the neighborhood of critical points (after ref. 8).

If simple pure imaginary characteristic roots exist, there is at least one real positive constant ω^* such that

$$L(i\omega^*) = H_1(i\omega^*) - H_2(i\omega^*)e^{-i\theta_k^*\omega^*} = 0 \tag{6}$$

and

$$\left.\frac{dL(s)}{ds}\right|_{s=i\omega^*} \neq 0$$

where the superscript $*$ indicates critical values as before. Equation (6) may be divided by $H_2(i\omega^*)$. If $H_2(i\omega^*) = 0$, then $H_1(i\omega^*) = 0$ and the stability of the system does not depend on θ_k; this case is not of interest here. Thus

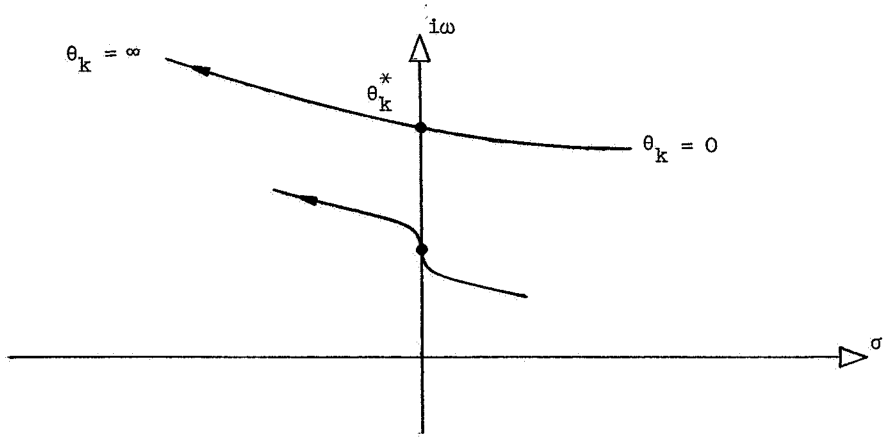
$$\frac{H_1(i\omega^*)}{H_2(i\omega^*)} = e^{-i\theta_k^*\omega^*} \tag{7}$$

From equation (7), the critical values of θ_k satisfy

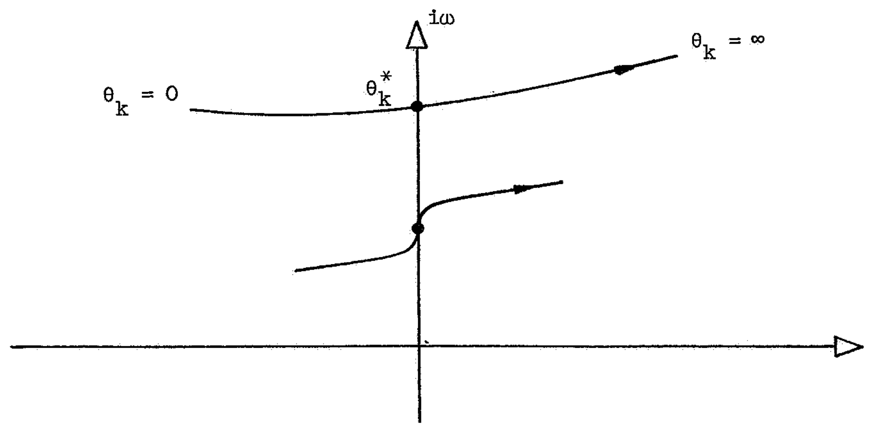
$$\left|\frac{H_1(i\omega^*)}{H_2(i\omega^*)}\right| = 1 \qquad \theta_k^* = -\frac{1}{\omega^*} \arg \frac{H_1(i\omega^*)}{H_2(i\omega^*)} \tag{8}$$

In general, the function $H_1(i\omega^*)$ will contain trigonometric functions of ω^*, and equation (8) will not necessarily yield an explicit expression for ω^*. If an explicit expression for ω^* cannot be obtained, an iteration technique will locate the ω^* values. In general the maximum value of ω^* for the iteration will have to be determined separately. The number of critical values of θ_k is zero or countably infinite, as can be seen in equation (8), since the various possible values of $\theta_k^*\omega^*$ differ by integral multiples of 2π. If the number of critical values is zero, the delay θ_k does not affect the stability of the system, since the absence of critical values means that no roots lie on the imaginary axis for that range of θ_k. If there are an infinite number of intersections, the way in which the loci intersect the imaginary axis for different critical values of θ_k must be found.

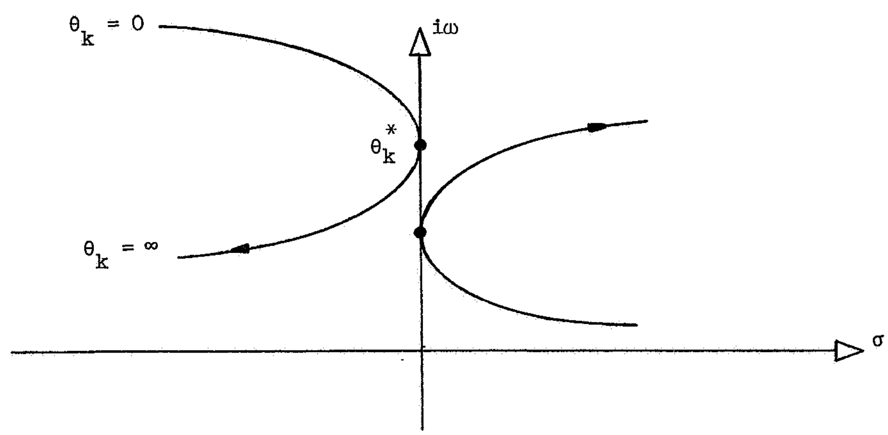
There are three possible ways a root locus can intersect the imaginary axis. These cases are illustrated in figure 2. In case (a) the loci cut from the right half-plane to the left half-plane. In case (b) the loci cut from left to right. In case (c) the loci do not cut across the imaginary axis but do touch it.



(a) Loci move from right to left as θ_k increases.



(b) Loci move from left to right as θ_k increases.



(c) Loci do not cross the imaginary axis.

Figure 2.- Possible root loci at critical points (after ref. 8).

The problem of finding how the root loci intersect the imaginary axis is now considered. Since θ_k is taken by itself, the results of reference 8 may be extended to this analysis as follows. The behavior of the root loci at critical points (i.e., how they intersect the imaginary axis) can be determined by finding both the sign of the derivative $\left.\frac{d^{M_k}\sigma}{d\theta_k^{M_k}}\right|_{\theta_k=\theta_k^*} \neq 0$ and the value of M_k, where M_k is defined as

$$M_k \equiv \min\left[m \left| \sigma^{(m)}\left(\theta_k^*\right) \neq 0\right.\right] \tag{9}$$

or $\sigma^{(m)}\left(\theta_k^*\right) = 0$ for $m = 1, 2, \ldots, M_k - 1$ and $\sigma^{(M_k)}\left(\theta_k^*\right) \neq 0$. (The superscript in parentheses denotes the mth derivative.) If a root locus cuts across the imaginary axis from left to right, then

$$\left.\begin{aligned} &\operatorname{sgn} \sigma^{(M_k)}\left(\theta_k^*\right) = +1 \\ \text{and} \quad & \\ &M_k = \text{Odd number} \end{aligned}\right\} \tag{10a}$$

If a root locus cuts across the imaginary axis from right to left, then

$$\left.\begin{aligned} &\operatorname{sgn} \sigma^{(M_k)}\left(\theta_k^*\right) = -1 \\ \text{and} \quad & \\ &M_k = \text{Odd number} \end{aligned}\right\} \tag{10b}$$

Finally, if a locus intersects but does not cut across the imaginary axis,

$$M_k = \text{Even number} \tag{10c}$$

The quantity $\sigma^{(M_k)}\left(\theta_k^*\right)$ is determined as follows. Substitution of $s = \sigma + i\omega$ into equation (5) yields

$$\begin{aligned} L(\sigma+i\omega) &= H_1(\sigma+i\omega) - H_2(\sigma+i\omega)e^{-\theta_k(\sigma+i\omega)} \\ &= L_R(\sigma,\omega,\theta_k) + iL_I(\sigma,\omega,\theta_k) \\ &= 0 \end{aligned} \tag{11}$$

Then it is clear that the root loci satisfy

$$L_R(\sigma,\omega,\theta_k) = 0 \tag{12}$$

$$L_I(\sigma,\omega,\theta_k) = 0 \tag{13}$$

In order to find the derivative of σ with respect to θ_k, equations (12) and (13) are differentiated with respect to θ_k to yield

$$\left.\begin{aligned}
\frac{dL_R}{d\theta_k} &= \frac{\partial L_R}{\partial \theta_k} + \frac{\partial L_R}{\partial \sigma}\frac{d\sigma}{d\theta_k} + \frac{\partial L_R}{\partial \omega}\frac{d\omega}{d\theta_k} = 0 \\
\frac{dL_I}{d\theta_k} &= \frac{\partial L_I}{\partial \theta_k} + \frac{\partial L_I}{\partial \sigma}\frac{d\sigma}{d\theta_k} + \frac{\partial L_I}{\partial \omega}\frac{d\omega}{d\theta_k} = 0
\end{aligned}\right\} \tag{14}$$

In terms of the Jacobian determinant,

$$\frac{d\sigma}{d\theta_k} = -\frac{1}{\Delta}\frac{\partial\left(L_R,L_I\right)}{\partial\left(\theta_k,\omega\right)} \tag{15}$$

and

$$\frac{d\omega}{d\theta_k} = -\frac{1}{\Delta}\frac{\partial\left(L_R,L_I\right)}{\partial\left(\sigma,\theta_k\right)} \tag{16}$$

where

$$\Delta \equiv \frac{\partial\left(L_R,L_I\right)}{\partial(\sigma,\omega)}$$

$$\frac{\partial\left(L_R,L_I\right)}{\partial(\sigma,\omega)} \equiv \begin{vmatrix} \dfrac{\partial L_R}{\partial \sigma} & \dfrac{\partial L_R}{\partial \omega} \\ \dfrac{\partial L_I}{\partial \sigma} & \dfrac{\partial L_I}{\partial \omega} \end{vmatrix} > 0 \tag{17}$$

$$\frac{\partial\left(L_R,L_I\right)}{\partial\left(\theta_k,\omega\right)} \equiv \begin{vmatrix} \dfrac{\partial L_R}{\partial \theta_k} & \dfrac{\partial L_R}{\partial \omega} \\ \\ \dfrac{\partial L_I}{\partial \theta_k} & \dfrac{\partial L_I}{\partial \omega} \end{vmatrix} \tag{18}$$

and

$$\frac{\partial\left(L_R,L_I\right)}{\partial\left(\sigma,\theta_k\right)} \equiv \begin{vmatrix} \dfrac{\partial L_R}{\partial \sigma} & \dfrac{\partial L_R}{\partial \theta_k} \\ \\ \dfrac{\partial L_I}{\partial \sigma} & \dfrac{\partial L_I}{\partial \theta_k} \end{vmatrix} \tag{19}$$

In reference 8 it is shown that the inequality in equation (17) will hold as long as the problem is well defined; that is, if a_N is not equal to zero and $b_n\theta_n$ $(n = 0, 1, \ldots, N - 1)$ are not all equal to zero. Thus, except for the case of branch points (referred to as critical points in complex variable theory) where $\left.\frac{dL(s)}{ds}\right|_{s=i\omega^*} = 0$, the derivatives $\frac{d\sigma}{d\theta_k}$ and $\frac{d\omega}{d\theta_k}$ given by equations (15) and (16) will exist.

If $\frac{d\sigma}{d\theta_k} = 0$ at $\theta_k = \theta_k^*$ (note vertical tangent to root loci at critical points in figs. 2(a) and 2(b)), then $M_k > 1$ and higher derivatives must be found in order to describe the root locus behavior at critical points. These higher derivatives are found from the following expressions:

$$\frac{d^m\sigma}{d\theta_k^m} = \frac{\partial}{\partial\theta_k}\left(\frac{d^{m-1}\sigma}{d\theta_k^{m-1}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{d^{m-1}\sigma}{d\theta_k^{m-1}}\right)\frac{d\sigma}{d\theta_k} + \frac{\partial}{\partial\omega}\left(\frac{d^{m-1}\sigma}{d\theta_k^{m-1}}\right)\frac{d\omega}{d\theta_k} \tag{20}$$

$$\frac{d^m\omega}{d\theta_k^m} = \frac{\partial}{\partial\theta_k}\left(\frac{d^{m-1}\omega}{d\theta_k^{m-1}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{d^{m-1}\omega}{d\theta_k^{m-1}}\right)\frac{d\sigma}{d\theta_k} + \frac{\partial}{\partial\omega}\left(\frac{d^{m-1}\omega}{d\theta_k^{m-1}}\right)\frac{d\omega}{d\theta_k} \tag{21}$$

for $m > 1$.

Thus the value of M_k and the sign of the derivative $\sigma^{(M_k)}\left(\theta_k^*\right)$ which are necessary for determining the behavior of the root loci at critical points may now be found by using equations (9), (15), and (20). The critical values of θ_k are found by using equations (8), as previously discussed.

STABILITY INDICATIVE FUNCTION FOR LINEAR SYSTEMS WITH MULTIPLE DELAYS

By using the results of the preceding analysis, the stability indicative function can now be formulated for linear systems with multiple delays whose characteristic equation has the form of equation (4), repeated here for convenience:

$$L(s) = \sum_{n=0}^{N} a_n s^n + \sum_{n=0}^{N-1} b_n s^n e^{-\theta_n s} = 0 \qquad \left(a_N \neq 0\right)$$

It is noted that theorem 2 of the appendix insures that the stability characteristics of the system $L(s)\Big|_{\theta_k=0} = 0$ are, in the limiting case, those of $L(s) = 0$ for positive values of $\theta_k \ll 1$. If values of θ_n (where $n = 0, \ldots, N - 1$, and $n \neq k$) are fixed and θ_k is a parameter, and if the number of roots in the right half-plane for equation (4) with $\theta_k = 0$ is known, then the number of roots in the right half-plane for equation (4) can be determined as a function of θ_k. This follows from theorem 2 together with the fact that the root loci are continuous. The remaining problem is that, in general, the number of roots of $L(s)\Big|_{\theta_k=0} = 0$ that lie in the right half-plane is not known for the case of multiple delays. A technique is now developed for determining this number.

Initially all delays are set equal to zero, and the number of roots of the resulting Nth-degree polynomial having positive real parts and nonnegative imaginary parts is readily found. At this point one parameter, say θ_0, is introduced and is allowed to vary from zero to its prescribed value $\overline{\theta}_0$. If critical values of θ_0 are found (by using eqs. (8)) in the interval $0 \leqq \theta_0 \leqq \overline{\theta}_0$, say $\theta_{0,1}^* < \theta_{0,2}^* < \ldots < \overline{\theta}_0$, then starting with the smallest value, $\theta_{0,1}^*$, $M_{0,1}$ and $\operatorname{sgn} \sigma^{\left(M_{0,1}\right)}\left(\theta_{0,1}^*\right)$ are determined as previously discussed. Then, by using equations (10), it is determined whether a locus of characteristic roots of this system goes into or comes out of the left half-plane as θ_0 is increased beyond $\theta_{0,1}^*$. Because of symmetry, only the upper half-plane need be considered. The next critical value, $\theta_{0,2}^*$, is considered, and $M_{0,2}$ and sgn $\sigma^{\left(M_{0,2}\right)}\left(\theta_{0,2}^*\right)$ are found as before. Again equations (10) are used to determine the behavior of the root locus at $\theta_0 = \theta_{0,2}^*$. After this procedure, it will be known precisely how many roots lie in the upper right half-plane when $\theta_0 = \overline{\theta}_0$. The next parameter, say θ_1, is considered with $\theta_0 = \overline{\theta}_0$ and all other delays equal to zero. Now θ_1 is allowed to vary from zero to $\overline{\theta}_1$ and each critical value is taken account of as before, so that when $\theta_1 = \overline{\theta}_1$, the number of roots in the upper right half-plane will be known. This procedure is continued until only θ_k remains.

At this point, the number of roots of $L(s)\Big|_{\theta_k=0} = 0$ that have positive real parts and nonnegative imaginary parts (i.e., that lie in the upper right half-plane) will be known. The stability characteristics of equation (4) can now be expressed in terms of θ_k. To do this, θ_k is allowed to vary from zero to its maximum value, taking account of each critical value and finding M_k and sgn $\sigma^{\left(M_k\right)}\left(\theta_k^*\right)$ as above. If there are no roots of $L(s)\Big|_{\theta_k=0} = 0$ in the first quadrant of the s-plane, then by theorem 1 the system is stable when $\theta_k = 0$. The system remains stable as θ_k increases from zero until the first critical value is reached where M_k is odd and $\sigma^{\left(M_k\right)}\left(\theta_k^*\right)$ is positive; beyond this point the system becomes unstable. If more roots pass into the first quadrant as θ_k is increased further, an account of the number is kept. The system remains unstable until all these roots leave the first quadrant as θ_k becomes larger still. On the other hand, if some of the roots of $L(s)\Big|_{\theta_k=0} = 0$ do lie in the first quadrant, then by theorem 1 the system is unstable for $\theta_k = 0$. As before, the system remains unstable so long as there are any roots having positive real parts and nonnegative imaginary parts. Thus by keeping account of the number of roots in the first quadrant as θ_k is increased from zero, the stability characteristics of the system are determined in terms of the parameter θ_k.

This concept is now formulated into the "stability indicative function." As indicated above, all delays are set equal to zero in equation (4) to form the Nth-degree polynomial

$$\overline{L}(s) = L(s)\Big|_{\theta_{N-1}=\theta_{N-2}=\cdots=\theta_1=\theta_0=0} = 0$$

$$= a_N s^N + \sum_{n=0}^{N-1}\left(a_n + b_n\right)s^n = 0 \tag{22}$$

Then z, the number of roots of $\overline{L}(s) = 0$ that lie in the first quadrant of the s-plane, may be found by classical techniques. Without loss of generality it is assumed that $\overline{\theta}_0 \neq 0$. Then $\theta_1, \ldots, \theta_{N-1}$ are set equal to zero to form

$$L_0(s) = L(s)\Big|_{\theta_{N-1}=\theta_{N-2}=\ldots=\theta_2=\theta_1=0} = 0 \tag{23}$$

Next z_0, the number of roots of $L_0(s) = 0$ that lie in the first quadrant when $\theta_0 = \overline{\theta}_0$, is found. Since at each step in this procedure the number of roots in the first quadrant will be known initially, and since the loci are continuous as mentioned above, the "stability indicative function" for systems with a single delay introduced in reference 8 may be used to find z_0, z_1, and so forth.

Now z_0 is equal to z plus any roots of $L_0(s) = 0$ that pass into the first quadrant of the s-plane minus any that leave as θ_0 goes from zero to $\overline{\theta}_0$. As explained earlier, the value of M_0 and the sign of $\sigma^{(M_0)}(\theta_0^*)$ together determine the behavior of the root loci at the critical points. Thus, the stability indicative function for θ_0 is given by

$$I(\theta_0) = z + \sum_{\theta_0^* \in \Theta_0} \frac{1 - (-1)^{M_0}}{2} \operatorname{sgn} \sigma^{(M_0)}\left(\theta_0^*\right) \tag{24}$$

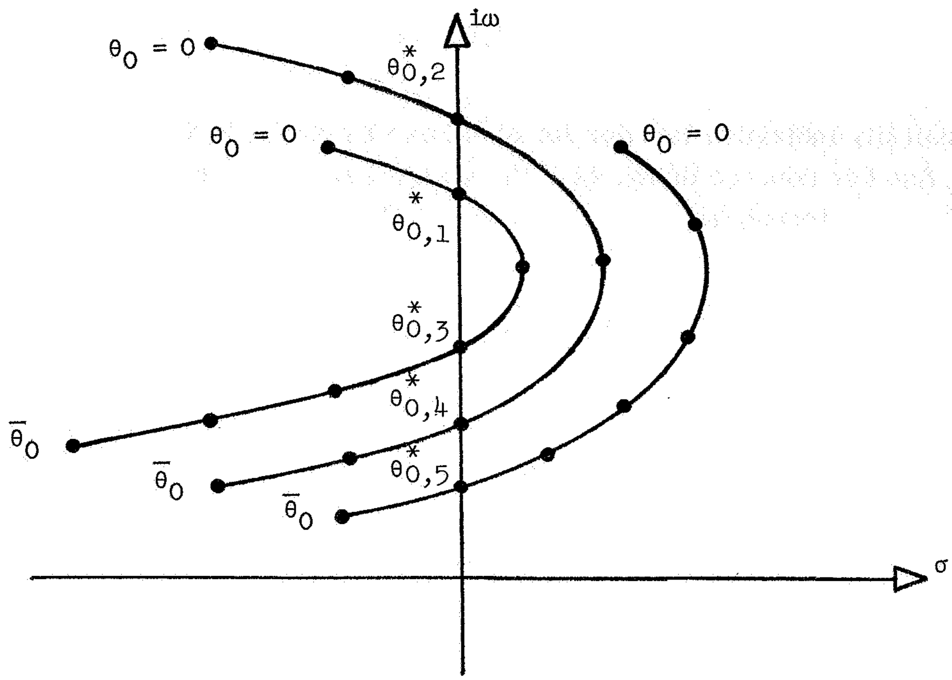
where

$$\Theta_0 = \left[\theta_0^* \middle| 0 \leqq \theta_0^* \leqq \theta_0\right] \tag{25}$$

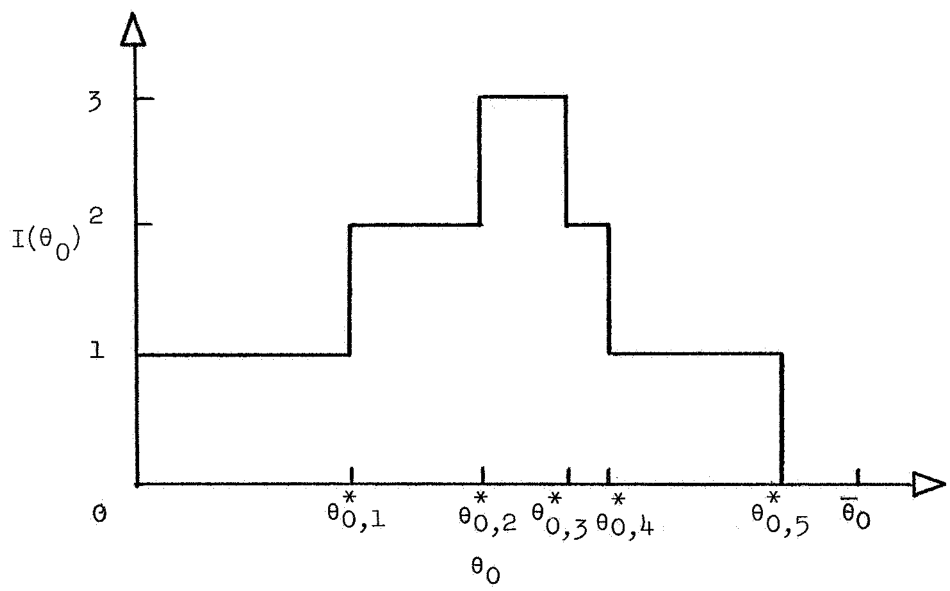
so that

$$z_0 = I\left(\overline{\theta}_0\right) \tag{26}$$

In order to clarify further the meaning of $I(\theta_0)$ and $I(\overline{\theta}_0)$, the following example is considered. It is assumed that there exists a system having $z = 1$ and the first five critical values of θ_0 such that $0 < \theta_{0,1}^* < \theta_{0,2}^* < \theta_{0,3}^* < \theta_{0,4}^* < \theta_{0,5}^* < \overline{\theta}_0$. If the behavior of the root loci is described by figure 3(a), then the function $I(\theta_0)$ would have the form shown in figure 3(b). It is clear that $I(0) = z = 1$ and $I(\overline{\theta}_0) = z_0 = 0$. Thus it is seen that the delay $\overline{\theta}_0$ has made this normally unstable system stable. In this example the root loci are shown only for clarity; it must be remembered that in applying the stability indicative function no attempt need be made to plot the root loci. Instead, the stability



(a) Behavior of the root loci for $0 \leqq \theta_0 \leqq \bar{\theta}_0$.



(b) Stability indicative function in terms of θ_0.

Figure 3.- Critical values of θ_0 and the stability indicative function $I(\theta_0)$.

indicative function is the only quantity of interest and may be plotted or tabulated, for example.

The stability indicative function for systems with multiple delays can now be developed. Another nonzero delay, say θ_1 with prescribed value $\overline{\theta}_1$, is considered. Here $L_1(s) = 0$ is formed as

$$L_1(s) = L(s)\Big|_{\theta_{N-1}=\theta_{N-2}=\ldots=\theta_3=\theta_2=0,\ \theta_0=\overline{\theta}_0} = 0 \tag{27}$$

and z_1, the number of roots of $L_1(s) = 0$ that lie in the first quadrant of the s-plane, is found. The value of z_1 is determined from the stability indicative function for θ_1 given by

$$I(\theta_1) = z_0 + \sum_{\theta_1^* \in \Theta_1} \frac{1 - (-1)^{M_1}}{2} \operatorname{sgn} \sigma^{(M_1)}\left(\theta_1^*\right) \tag{28}$$

where

$$\Theta_1 = \left[\theta_1^* \middle| 0 \leqq \theta_1^* \leqq \theta_1\right] \tag{29}$$

so that

$$z_1 = I\left(\overline{\theta}_1\right) \tag{30}$$

The next delay, say θ_2, is considered. Here $L_2(s) = 0$ is formed with $\theta_0 = \overline{\theta}_0$ and $\theta_1 = \overline{\theta}_1$; and z_2, the number of roots of $L_2(s) = 0$ that lie in the first quadrant of the s-plane, is found by using the stability indicative function. This procedure is continued, considering one delay at a time, until θ_k, the given parameter, is the only one remaining.

At this point the number of roots of $L(s)\Big|_{\theta_k=0} = 0$ that have positive real parts and nonnegative imaginary parts is known. Then $\overline{z}$ is defined as the number of characteristic roots of $L(s)\Big|_{\theta_k=0} = 0$ whose real parts are positive and imaginary parts are nonnegative. Now $\overline{z}$ can be expressed in terms of the above results as

$$\bar{z} = z + \sum_{j=0}^{k-1} \sum_{\theta_j^* \in \Theta_j} \frac{1 - (-1)^{M_j}}{2} \operatorname{sgn} \sigma^{(M_j)}\left(\theta_j^*\right) + \sum_{j=k+1}^{N-1} \sum_{\theta_j^* \in \Theta_j} \frac{1 - (-1)^{M_j}}{2} \operatorname{sgn} \sigma^{(M_j)}\left(\theta_j^*\right) \tag{31}$$

Stability criterion.- If the stability indicative function for the parameter θ_k is constructed as

$$I(\theta_k) = \bar{z} + \sum_{\theta_k^* \in \Theta_k} \frac{1 - (-1)^{M_k}}{2} \operatorname{sgn} \sigma^{(M_k)}\left(\theta_k^*\right) \tag{32}$$

then the system whose transcendental characteristic equation is $L(s) = 0$ (eq. (4)) is stable if

$$I(\theta_k) = 0 \tag{33}$$

and unstable if

$$I(\theta_k) > 0 \tag{34}$$

Proof: By using equations (10) together with theorem 2 of the appendix, it is shown that the function $I(\theta_k)$ given in equation (32) indicates the number of characteristic roots of equation (4) whose real parts are positive and imaginary parts are nonnegative for a given value of θ_k. Hence, from theorem 1 of the appendix, if $I(\theta_k) = 0$ the system is stable, and if $I(\theta_k) > 0$ the system is unstable.

STABILITY OF A SECOND-ORDER LINEAR SYSTEM WITH MULTIPLE DELAYS

In order to illustrate the application of the stability indicative function for linear systems with multiple delays, the following differential-difference equation is considered:

$$\frac{d^2x(\tau)}{d\tau^2} + 2\zeta\Omega \frac{dx(\tau - T_V)}{d\tau} + \Omega^2 x\left(\tau - T_p\right) = 0 \tag{35}$$

The position delay T_p is assumed to have a prescribed value and the stability boundaries of this system are to be found in terms of the velocity delay T_V and the parameter ζ. To be compatible with the results of reference 8, which includes the stability boundaries of equation (35) for the three cases where $T_V = T_p$, $T_V = 0$, and $T_p = 0$, equation (35) is normalized as follows:

Let

$$t = \Omega\tau \qquad \theta_0 = T_p\Omega \qquad \theta_1 = T_V\Omega \tag{36}$$

Then equation (35) becomes

$$\frac{d^2x(t)}{dt^2} + 2\zeta \frac{dx(t - \theta_1)}{dt} + x(t - \theta_0) = 0 \tag{37}$$

The corresponding characteristic equation is immediately obtained as

$$L(s) = s^2 + 2\zeta s e^{-\theta_1 s} + e^{-\theta_0 s} = 0 \tag{38}$$

According to the theory, the first step in finding the stability characteristics of this system is to set the delays θ_0 and θ_1 equal to zero and find the roots of the resulting second-degree polynomial. With $\theta_0 = \theta_1 = 0$ there results

$$\overline{L}(s) = s^2 + 2\zeta s + 1 = 0 \tag{39}$$

whose characteristic roots are

$$s = -\zeta \pm i\sqrt{1 - \zeta^2} \tag{40}$$

It is clear that for $\zeta > 0$ there is no root in the first quadrant of the s-plane. Thus, $z = 0$ for $\zeta \geqq 0$.

Since θ_1 is the normalized parameter of interest, the next step is to set θ_1 equal to 0 and obtain

$$L_0(s) = s^2 + 2\zeta s + e^{-\theta_0 s} = 0 \tag{41}$$

By using equations (8) the critical values of θ_0 and ω for equation (41) are found to be

$$\omega^* = \sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}} \tag{42}$$

$$\theta_0^* = \frac{1}{\omega^*}\left[\left(\tan^{-1} \frac{2\zeta}{\omega^*}\right) + 2p\pi\right] \qquad (p = 0, 1, 2, \ldots) \tag{43}$$

The value of M_0 and the sign of the derivative $\sigma^{(M_0)}\left(\theta_0^*\right)$ are now found by using equations (9), (15), (16), and (20). From equations (12), (13), and (41),

$$\left.\begin{aligned} L_R(\sigma,\omega,\theta_0) &= \sigma^2 - \omega^2 + 2\zeta\sigma + e^{-\theta_0\sigma}\cos\theta_0\omega \\ L_I(\sigma,\omega,\theta_0) &= 2\omega(\sigma+\zeta) - e^{-\theta_0\sigma}\sin\theta_0\omega \end{aligned}\right\} \tag{44}$$

Then

$$\left.\frac{d\sigma}{d\theta_0}\right|_{\theta_0=\theta_0^*} = -\frac{1}{\Delta}\left.\frac{\partial\left(L_R,L_I\right)}{\partial(\theta_0,\omega)}\right|_{\theta_0=\theta_0^*}$$

$$= -\frac{1}{\Delta}\begin{vmatrix} -2\zeta(\omega^*)^2 & -2\omega^* - 2\zeta\omega^*\theta_0^* \\ -(\omega^*)^3 & 2\zeta - \theta_0^*(\omega^*)^2 \end{vmatrix}$$

$$= \frac{(\omega^*)^2}{\Delta}\left[4\zeta^2 + 2(\omega^*)^2\right] \tag{45}$$

Clearly

$$\left.\frac{d\sigma}{d\theta_0}\right|_{\theta_0=\theta_0^*} > 0$$

so that $M_0 = 1$ and $\operatorname{sgn}\sigma^{(M_0)}\left(\theta_0^*\right) = +1$. Then for $\zeta > 0$ it is found from equations (24) and (26) that

$$z_0 = 0$$

for

$$0 \leqq \theta_0 \leqq \frac{1}{\omega^*}\tan^{-1}\frac{2\zeta}{\omega^*}$$

and

$$z_0 = p$$

for

$$\frac{1}{\omega^*}\left[\left(\tan^{-1}\frac{2\zeta}{\omega^*}\right) + 2(p-1)\pi\right] < \theta_0 \leq \frac{1}{\omega^*}\left[\left(\tan^{-1}\frac{2\zeta}{\omega^*}\right) + 2p\pi\right] \qquad (p = 1, 2, . . .)$$

Now the parameter θ_1 is considered, with $\theta_0 = \overline{\theta}_0 = T_p\Omega$. Equation (38) becomes

$$L_1(s) = s^2 + 2\zeta s e^{-\theta_1 s} + e^{-\overline{\theta}_0 s} = 0 \qquad (46)$$

By using equations (8), the expressions for the critical values of θ_1 and ω are found as

$$(\omega^*)^4 - 4\zeta^2(\omega^*)^2 - 2(\omega^*)^2\cos\overline{\theta}_0\omega^* + 1 = 0 \qquad (47)$$

and

$$\theta_1^* = \frac{1}{\omega^*}\left\{\left[\tan^{-1}\frac{(\omega^*)^2 - \cos\overline{\theta}_0\omega^*}{\sin\overline{\theta}_0\omega^*}\right] + 2p\pi\right\} \qquad (p = 0, 1, . . .) \qquad (48)$$

The critical values of ω^* are found for different values of ζ by iterating equation (47) from $\omega = \omega_{min}$ to $\omega = \omega_{max}$. Approximate values for ω_{min} and ω_{max} are found from equation (47) by solving for $\cos\overline{\theta}_0\omega^*$ and finding the intersections of

$$f_1(\omega) = \cos\overline{\theta}_0\omega \qquad (49)$$

with

$$f_2(\omega) = \frac{\omega^4 - 4\zeta^2\omega^2 + 1}{2\omega^2} \qquad (50)$$

by a graphical technique.

The value of M_1 and the sign of the derivative $\sigma^{(M_1)}(\theta_1^*)$ are now determined. From equation (46),

$$L_R(\sigma,\omega,\theta_1) = \sigma^2 - \omega^2 + 2\zeta\sigma e^{-\theta_1\sigma}\cos\theta_1\omega + 2\zeta\omega e^{-\theta_1\sigma}\sin\theta_1\omega + e^{-\overline{\theta}_0\sigma}\cos\overline{\theta}_0\omega \qquad (51)$$

and

$$L_I(\sigma,\omega,\theta_1) = 2\sigma\omega - 2\zeta\sigma e^{-\theta_1\sigma}\sin\theta_1\omega + 2\zeta\omega e^{-\theta_1\sigma}\cos\theta_1\omega - e^{-\overline{\theta}_0\sigma}\sin\overline{\theta}_0\omega \tag{52}$$

Then

$$\left.\frac{d\sigma}{d\theta_1}\right|_{\theta_1=\theta_1^*} = -\frac{1}{\Delta}\frac{\partial\left(L_R,L_I\right)}{\partial\left(\theta_1,\omega\right)}$$

$$= \frac{1}{\Delta}\left[(\omega^*)^4 + (\omega^*)^3\overline{\theta}_0 \sin\overline{\theta}_0\omega^* - 1\right] \tag{53}$$

Equation (53) is evaluated by using the given value of $\overline{\theta}_0$ and the critical values of ω^* found from equations (44).

The stability boundaries of the system described by equation (35) may now be found by setting the stability indicative function for θ_1 equal to zero, where

$$I(\theta_1) = z_0 + \sum_{\theta_1^* \in \Theta_1} \operatorname{sgn}\left.\frac{d\sigma}{d\theta_1}\right|_{\theta_1=\theta_1^*} \tag{54}$$

Here it is assumed that $M = 1$; if $\left.\frac{d\sigma}{d\theta_1}\right|_{\theta_1=\theta_1^*} = 0$, then $M \neq 1$ and higher derivatives must be found as discussed earlier.

Equations (47), (48), (49), (50), and (54) were solved for a range of values of $\overline{\theta}_0$ and ζ; in this example, for $\overline{\theta}_0$ between 0 and 2.6 and for ζ between 0 and 1.2. The results of these computations are presented in figures 4 to 8 in the (θ_1,ζ)-plane with $\overline{\theta}_0$ as a parameter.

Figure 4 gives the stability regions of equation (37) for $\overline{\theta}_0 = 0$. For small values of θ_1 the system is stable. For small values of ζ the stability of the system alternates as θ_1 is increased; however, at some value of θ_1 the system becomes unstable and it remains unstable for larger values of θ_1. Note that the stability regions continue to repeat but diminish in size as θ_1 increases without limit. These results correspond to those obtained in reference 8.

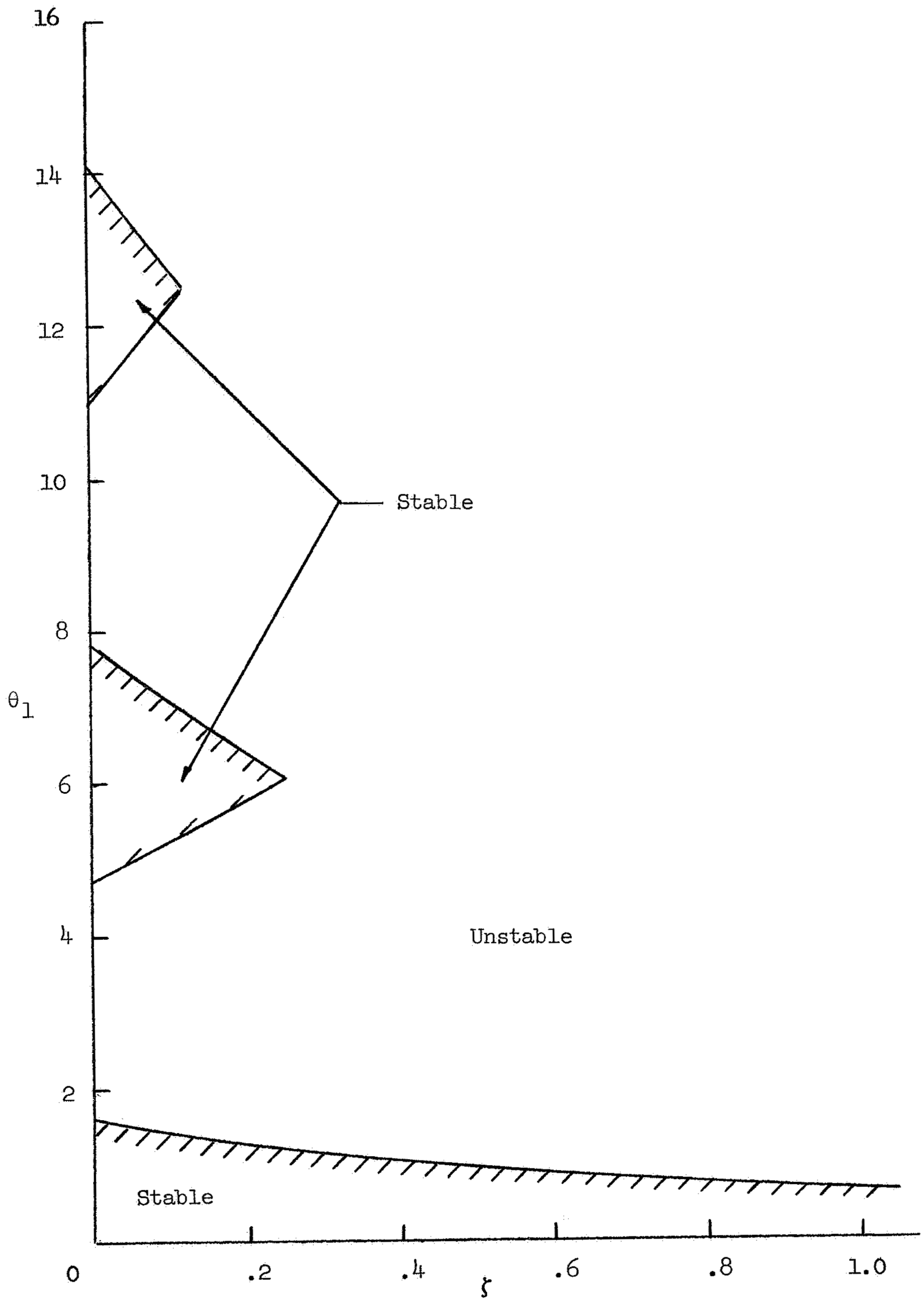


Figure 4.- Stability boundaries of the system $\frac{d^2x(t)}{dt^2} + 2\zeta \frac{dx\left(t - \theta_1\right)}{dt} + x\left(t - \bar{\theta}_0\right) = 0$ for $\bar{\theta}_0 = 0$.

Figure 5 shows the stability boundaries of equation (37) for $\bar{\theta}_0 = 0.05$. These boundaries are similar in some respects to those obtained when $\bar{\theta}_0 = 0$, but for values of ζ less than approximately 0.025, the system is unstable for all values of θ_1. It is seen that each region of stability for this case is located within each region of the previous case. However, the stability regions vanish for sufficiently large values of θ_1.

The results of this example are slightly more useful when presented in the form of a topological plot of the stability regions. This is done for the three regions of figure 4 in figures 6, 7, and 8. The coordinate scale factors are modified to allow enlarged plots. Figure 6 is probably of most practical value since the time delays will usually be small and will fall in the range of the variables considered. The area between each curve and the abscissa represents a region of stability for the particular value of $\bar{\theta}_0$ shown. It can be seen that when $\theta_1 = 1$ and $\zeta = 1$ the system is unstable for $0 \leqq \bar{\theta}_0 \leqq 1.8$ but becomes stable for $2.0 \leqq \bar{\theta}_0 \leqq 2.4$; that is, introduction of the second delay θ_0 has stabilized the normally unstable system. Figure 7 shows the destabilizing effects of the second delay θ_0 and indicates that the regions of stability vanish completely for $\bar{\theta}_0$ greater than approximately 0.3. Figure 8 shows a further reduction in the maximum value of $\bar{\theta}_0$ for which the system can be stable. Finally, it is noted that if the system is unstable for a given set of values in the $(\zeta,\theta_1,\bar{\theta}_0)$-space, say (0.1,4.0,0.1), the unstable system can be made stable by increasing θ_1 to about 6; but increasing θ_1 to the range shown in figure 8 or beyond will not stabilize the system.

CONCLUDING REMARKS

A theoretical study has been made of the stability of a class of linear differential-difference equations with arbitrary multiple delays. A direct method for determining the exact stability boundaries for homogeneous, linear differential-difference equations with constant coefficients and constant delays was formulated. The formulation resulted in a stability indicative function depending on a single parameter. This function determines the number of roots of the transcendental characteristic equation that have positive real parts for a given set of coefficients and delays, and it is proved that the system is stable if, and only if, this function has a value of zero.

The stability indicative function for such systems compared favorably, in terms of range of application and ease of implementation, with other stability criteria. For example, Pontryagin's criterion is limited to systems with integral multiple delays and the Lyapunov method generally does not give the largest region of stability without an exhaustive search for the proper Lyapunov function. A second-order system with delays in the velocity and position feedback terms was considered as an example, and the stability regions for this system were determined for a range of delays and coefficients. It was observed that introduction of a delay has a definite destabilizing effect on the system, and

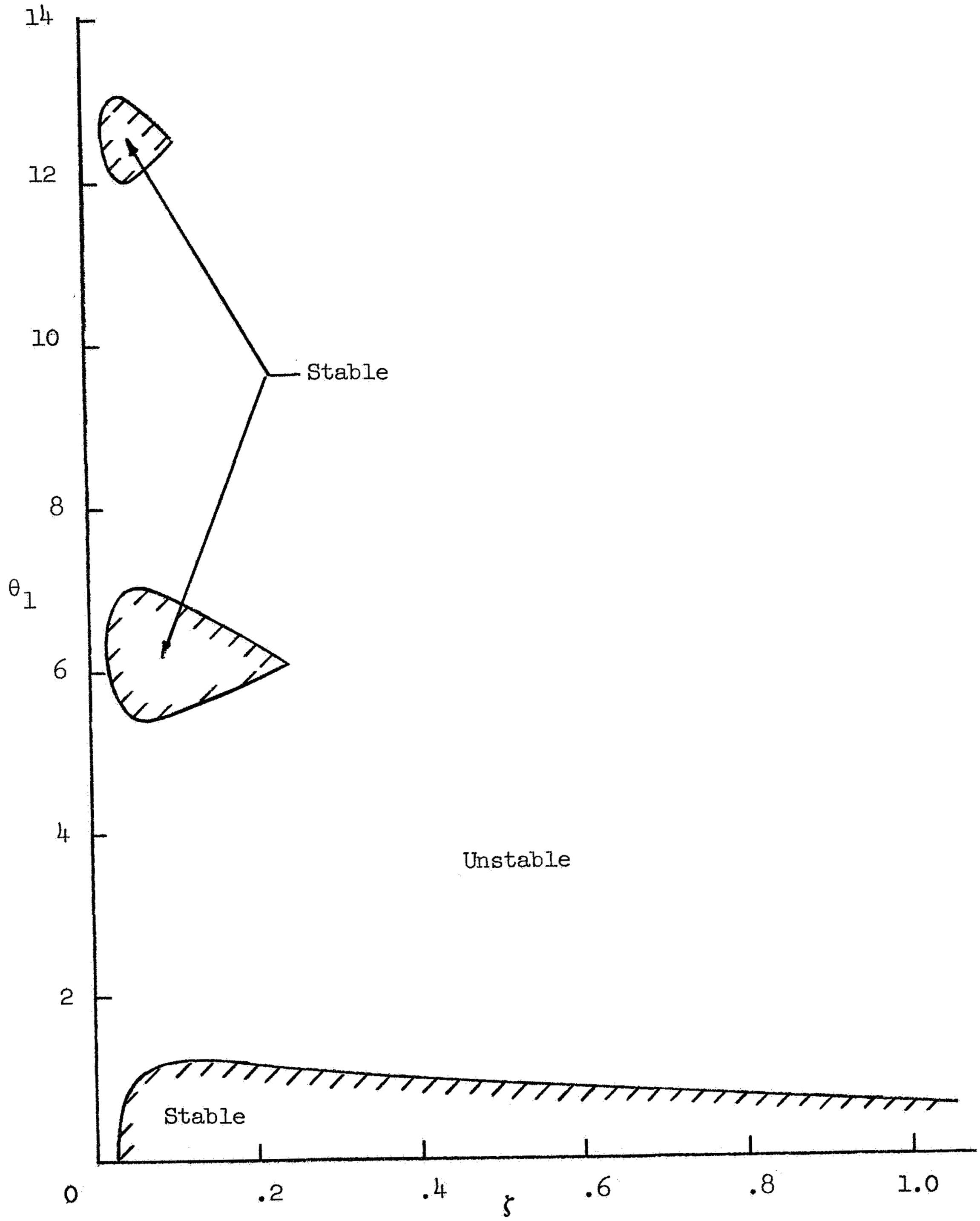


Figure 5.- Stability boundaries of the system $\ddot{x}(t) + 2\zeta\dot{x}\left(t - \theta_1\right) + x\left(t - \bar{\theta}_0\right) = 0$ for $\bar{\theta}_0 = 0.05$.

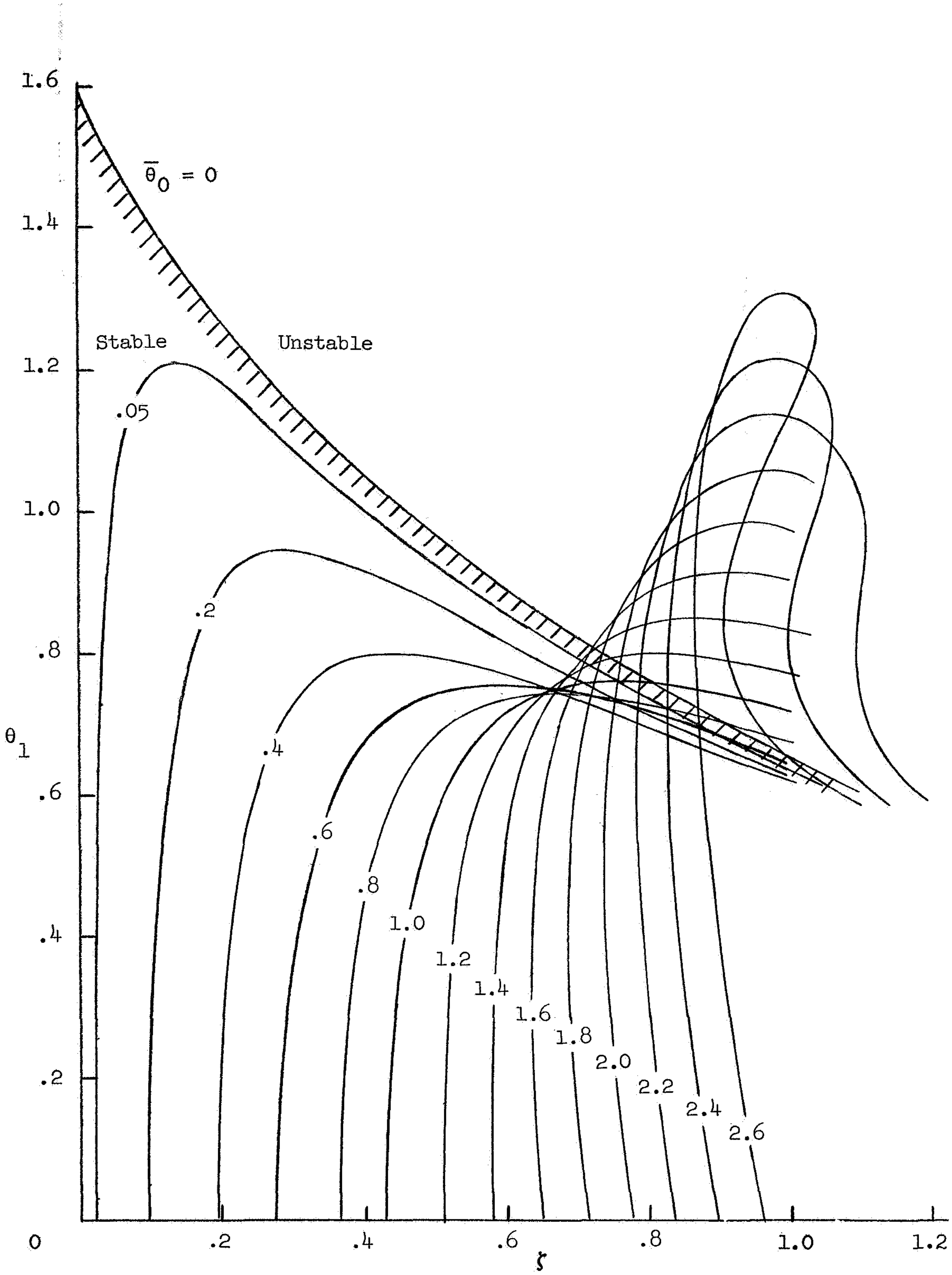


Figure 6.- Stability boundaries of the system $\ddot{x}(t) + 2\zeta\dot{x}(t - \theta_1) + x(t - \bar{\theta}_0) = 0$ for several values of $\bar{\theta}_0$ with θ_1 between 0 and 1.6.

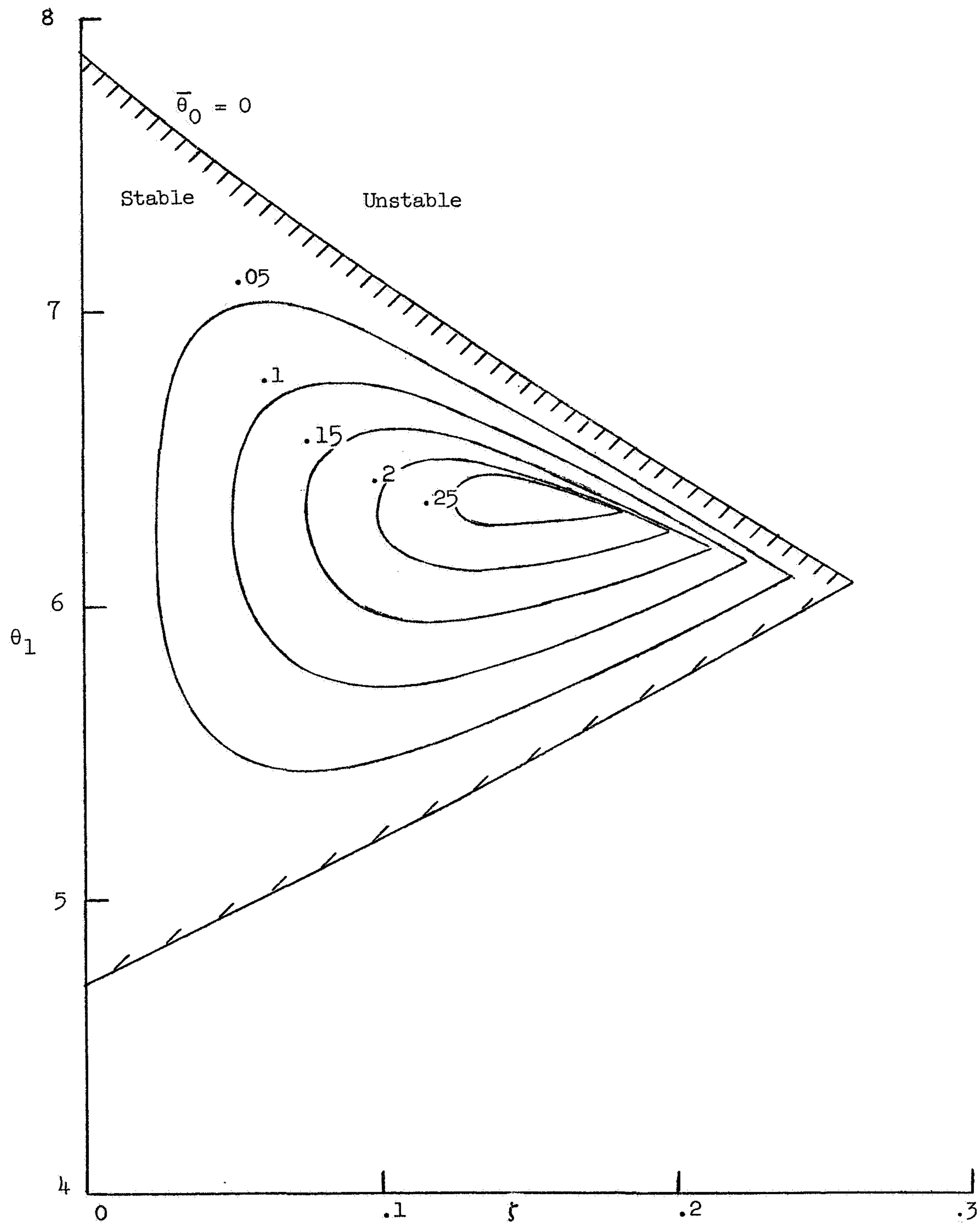


Figure 7.- Stability boundaries of the system $\ddot{x}(t) + 2\zeta\dot{x}\left(t - \theta_1\right) + x\left(t - \bar{\theta}_0\right) = 0$ for several values of $\bar{\theta}_0$ with θ_1 between 4 and 8.

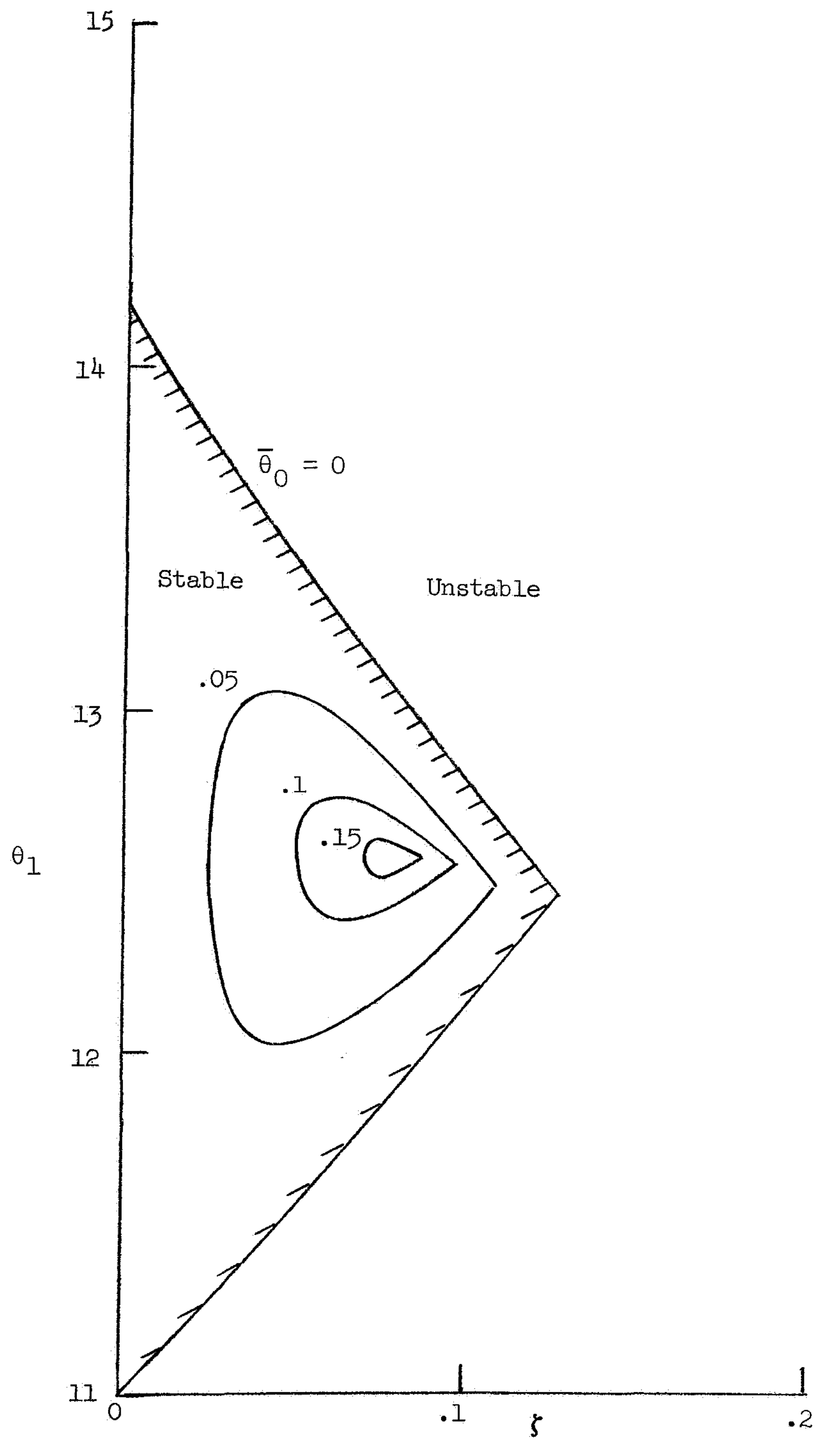


Figure 8.- Stability boundaries of the system $\ddot{x}(t) + 2\zeta\dot{x}(t - \theta_1) + x(t - \bar{\theta}_0) = 0$ for several values of $\bar{\theta}_0$ with θ_1 between 11 and 15.

introduction of a second delay has a compounding effect to further reduce stability. However, this example clearly illustrates that certain combinations of delays can stabilize an unstable system.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 20, 1968,
124-07-02-16-23.

APPENDIX

THEOREMS FOR THE DEVELOPMENT OF THE STABILITY INDICATIVE FUNCTION

Theorem 1

The homogeneous linear system whose characteristic equation is

$$L(s) = \sum_{n=0}^{N} a_n s^n + \sum_{n=0}^{N-1} b_n s^n e^{-\theta_n s} = 0 \qquad (a_N \neq 0)$$

(where the coefficients a_n and b_n are real constants and the constant delays θ_n are nonnegative finite values and are not necessarily distinct) is asymptotically stable if, and only if, all the characteristic roots have negative real parts.

Proof.- See reference 11.

Theorem 2

For a sufficiently small positive value of θ_k with $b_k \neq 0$, the finite roots of $L(s) = 0$ can be made arbitrarily close to the finite roots of $L(s)\big|_{\theta_k=0} = 0$, and there exist an infinite number of roots whose real parts are negatively infinite.

Proof.- Express the roots of $L(s) = 0$ as $s = \sigma + i\omega$, consider the following assumptions, and prove the theorem by contradiction.

Case 1: $\lim_{\theta_k \to 0} \sigma = \alpha$ and $\lim_{\theta_k \to 0} \omega = \beta$ where α and β are finite constants.

Thus,

$$\lim_{\theta_k \to 0} L(s) = \sum_{n=0}^{N} a_n s^n + b_k s^k + \sum_{n=0}^{k-1} b_n s^n e^{-\theta_n s} + \sum_{n=k+1}^{N-1} b_n s^n e^{-\theta_n s} = 0$$

Hence, we see that the roots of $L(s) = 0$ are close to the roots of $L(s)\big|_{\theta_k=0} = 0$ for small values of θ_k.

Case 2: $\lim_{\theta_k \to 0} \sigma = \alpha$ and $\lim_{\theta_k \to 0} \omega = \infty$ where α is a finite constant. Then

$$\lim_{\theta_k \to 0} \left| e^{-\theta_n s} \right| = \lim_{\theta_k \to 0} e^{-\theta_n \sigma} = e^{-\theta_n \alpha}$$

and

$$\lim_{\theta_k \to 0} \left| \frac{1}{s} \right| = \lim_{\theta_k \to 0} \frac{1}{\sqrt{\sigma^2 + \omega^2}} = 0$$

Divide both sides of $L(s) = 0$ by $s^N \neq 0$ to get

$$\sum_{n=0}^{N} a_n s^{n-N} + \sum_{n=0}^{N-1} b_n s^{n-N} e^{-\theta_n s} = 0$$

or

$$a_N + \sum_{n=0}^{N-1} \left(a_n + b_n e^{-\theta_n s} \right) s^{n-N} = 0$$

If case 2 holds, then each term in brackets is finite and $a_N = 0$. This result contradicts the definition of $L(s) = 0$; hence, case 2 does not hold.

Case 3: $\lim_{\theta_k \to 0} \sigma = +\infty$. Then

$$\lim_{\theta_k \to 0} \left| e^{-\theta_n s} \right| = \lim_{\theta_k \to 0} e^{-\theta_n \sigma} = 0$$

and

$$\lim_{\theta_k \to 0} \left| \frac{1}{s} \right| = \lim_{\theta_k \to 0} \frac{1}{\sqrt{\sigma^2 + \omega^2}} = 0$$

Again, this result requires that $a_N = 0$. Thus case 3 does not hold, and the only other possibility wherein $a_N \neq 0$ is that

$$\lim_{\theta_k \to 0} \sigma = -\infty$$

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